Section-B

Quantitative Techniques In Decision Making

Linear Programming 6

SLOB Mapped against the Module

To equip oneself with application-oriented knowledge of Linear Programming techniques to facilitate management decisions for optimisation through resource allocation, managing competition, work scheduling and managing cost overrun, demand estimation, production and cost analysis etc.

Module Learning Objectives:

After studying this module, the students will be able to:

- Describe the role of mathematical model in Decision Making.
- Understand the meaning and requirements of Linear Programming problems or constrained optimization models.
- Understand the advantages and limitations of Linear Programming.
- Understand the application of Linear Programming in different fields.
- Formulate the Linear Programming Problems.
- Solve the problems of Linear Programming using different methods.
- Understand the concept of Duality

Linear Programming

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With the advancement of computer technology use of mathematical models have increased substantially in different areas of business, manufacturing etc. A model represents the essential features of an object, system or a problem without unnecessary details. A mathematical model is one where the important aspects are represented in mathematical form using variables, parameters and functions. This facilitates to determine the best system design or action to take, as the case may be. Instead of constructing and manipulating real life systems we get a much cheaper, faster and safer solution by using mathematical models. If only limited resources are there then using mathematical model one can find how best those can be used to maximize the profit or minimize the cost. When an organization has a given amount of resources (man, machine, money, space etc.) at its disposal and also the output per unit of resource as well as the return per unit of output are known then the combination of resource to give maximum profit (or to have minimum cost) can be easily determined with the help of mathematical model. In fact the situations which require a search for best values of the variables, subject to certain restrictions are amenable to programming analysis. When a programming analysis problem has the features in which the total effectiveness is expressed as a linear function of individual allocations and the restrictions on resources give rise to linear equalities or inequalities of the individual allocations then the same is called Linear Programming Problem. These are also known as Constrained Optimization Problems due to the presence of restrictions or constraints under which the problem is solved.

Linear Programming is an optimization technique. It is "**a technique for specifying how to use limited resources or capacities of a business to obtain a particular objective, such as least cost, highest margin or least time, when those resources have alternate uses**".

A linear programming problem has three basic components.

- Decision Variables:- These are the physical quantities controlled by the decision maker and represented by mathematical symbols. As example we can say that x_i is the number of units of product i that can be produced by an organization in a particular month. Decision variables can take any one of a set of possible values.
- Objective Function:- This defines the criteria for evaluating the solution. It is a mathematical function of the decision variables. For example, the objective function may measure the profit or cost as a function of the quantities of various products produced by an organization. Also it specifies a direction of optimization – to maximize some return (e.g Profit) or to minimize some cost (e.g production cost, investment cost etc.)
- Constraints:- These are a set of functional inequalities or equalities that represent physical, economic, technological or other restrictions under which optimization is to be accomplished. For example, constraints might ensure that no more input is used than is available.

Definition of Linear Programming

According to Kohlar **"A method of planning and operation involved in the construction of a model of a real situation containing the following elements: (a) variables representing the available choices, and**

(b) mathematical expressions (i) relating the variables to the controlling conditions, and (ii) reflecting the criteria to be used in measuring the benefits derivable from each of the several possible plans, and (iii) establishing the objective. The method may be so devised as to ensure the selection of the best of a large number of alternatives ”.

Samuelson, Dorfman and Solow defines LP as **“The analysis of problems in which a linear function of a number of variables is to be maximized (or minimized) when those variables are subject to a number of restraints in the form of linear inequalities”**.

In the words of Loomba, **“LP is only one aspect of what has been called a system approach to management wherein all programmes are designed and evaluated in terms of their ultimate effects in the realization of business objectives”**.

Requirements of Linear Programming

1. Decision variables and their relationship:- The relationship among different Decision variables must be linear.
2. Well defined Objective Function:- Any LPP should have a well defined objective function to optimize which may either be to maximize contribution or be to minimize the cost by utilizing the available scarce resources.
3. Presence of constraints or restrictions:- There must be limitations on availability of resources expressed as linear equations or inequalities
4. Alternative courses of action:- It must be possible to make a selection between various combinations of the productive factors.
5. Non negativity constraints:- All the decision variables must assume non negative values because these variables represent physical quantities which cannot be negative.
6. Linearity:- Both the Objective Function and the constraints must be expressed as linear equation or inequality, as the case may be.
7. Finiteness:- The number of decision variables and constraints must be finite.
8. Additivity:- The sum of the resources used by different activities must be equal to the total quantity of resources individually and collectively.
9. Divisibility:- This implies that the solutions need not be in whole numbers. Rather they are divisible and may take any fractional value.
10. Deterministic:- Conditions of certainty is assumed to exist. In other words, the coefficients of the decision variables in both Objective Function and Constraints are completely known and do not change during the period of study i.e the coefficients are deterministic in nature.

Advantages of Linear Programming

Linear Programming has certain distinct advantages as given below.

1. Any LPP formulated properly gives a clear picture of the true problem. This is as valuable as the solution of the problem. In other words, insight and perspective into problem solution gets proper importance here.
2. Due consideration is given to all possible solutions of the problem. In real life, the management problems are so complex that a lot of difficulty is encountered in arriving at a feasible solution. But whenever Linear Programming technique is used an optimal solution is ensured irrespective of the degree of difficulty of the problem.
3. Better and more successful decisions can be made by using linear programming. Proper formulation of LPP

always show the limitations and restrictions under which one has to operate. Thus if it is necessary to deviate from the best solution then one can easily evaluate the quantum of penalty for the deviation.

4. Any plan arrived at through Linear Programming can always be reevaluated based on the changing conditions. Hence LP is a better tool for adjusting to meet the changing conditions.
5. It highlights any bottleneck in the production process.
6. It facilitates optimal use of productive factors as well as best use of existing facilities.
7. It provides flexibility in analyzing a variety of multidimensional problems.
8. An information base is created with its help which ultimately facilitates allocation of scarce resources.

Limitations of Linear Programming

Although linear programming is a very useful technique for solving optimization problems, there are certain important limitations in the application of linear programming. Some of these are discussed below:

1. The linear programming models can be applied only in those situations where the constraints and the objective function can be stated in terms of linear expressions. But in real life situations many objective functions and constraints cannot be expressed linearly.
2. In linear programming problems, coefficients of the decision variables in the objective function and the constraint must be completely known and they should not change during the period of study. In practice it may not be possible to know all the coefficients with certainty.
3. Linear programming may give fractional valued answers of the decision variables which in some cases (like number of a specific type of car to be produced) may be redundant.
4. Linear programming will fail to give a solution if management have conflicting multiple goals.
5. Linear programming problem requires that the total measure of effectiveness and total resource usage resulting from the joint performance of the activities must be equal to the respective sums of these quantities resulting from each activity being performed individually.
6. Many real-world problems are so complex, in terms of the number of variables and relationships constrained in them, that they tax the capacity of even the largest computer.
7. Other limitations of LP includes:-
   - Does not take into consideration the effect of time and uncertainty.
   - Parameters appearing in the model are assumed to be constants but in real-life situations they are frequently neither known nor constants.

Application Areas of Linear Programming

In practice linear programming has proved to be one of the most widely used technique of managerial decision making in business, industry and numerous other fields.

1. **Industrial Applications:**

   Linear programming is extensively used to solve a variety of industrial problems. In each of these applications, the general objective is to determine a plan for production and procurement in the time period under consideration. It is necessary to satisfy all demand requirements without violating any of the constraints. Few examples of industrial applications are as follows:

   (a) Product Mix-Problem.

(b) Production Scheduling.

(c) Production Smoothing Problem

(d) Blending Problem

(e) Transportation Problem

(f) Product Distribution Problem.

(g) Linear programming is also used by oil refineries to determine the optimal mix of products to be produced by the refinery during a given period..

(h) Communication Industry: LP methods are used in solving problems involving facilities for transmission, switching, relaying etc.

(i) Rail Road Industry: An LP model for optimal programming of railway freight, and train movements has been formulated to handle scheduling problems as found at large terminal switching rail points.

2. Management Applications:

(a) Portfolio Selection.

(b) Financial Mix Strategy.

(c) Profit Planning.

(d) Media Selection.

(e) Travelling Salesmen Problem.

(f) Determination of equitable salaries.

(g) Staffing problem.

3. Miscellaneous Applications:

The additional application of Linear Programming are as follows:

(a) Farm planning.

- The particular crops to be grown or cattle to keep during a period
- The acreage to be devoted to each, and
- The particular production methods to be used.

(b) Airline routing.

(c) Administration, Education and Politics have also employed linear programming to solve their problems.

(d) Diet Problems: The diet problem, one of the earliest applications of linear programming was originally used by hospitals to determine the most economical diet for patients.

4. Administrative applications:

Linear programming can be used for administrative applications. Administrative applications of Linear Programming are concerned with optimal usage of resources like men, machine and material.

5. Non-Industrial applications:

Linear programming techniques/tools can be applied in the case of non-industrial applications as well. Examples of the use of L.P. techniques for non-industrial applications are given below:

- Agriculture.
- Environmental Protection.

- Urban Department.
- Facilities Location.

6. Further applications of Linear Programming are:

(i) In structural design for maximum product.

(ii) In balancing assembly lines.

(iii) In scheduling of a military tanker fleet.

(iv) In determining which parts to make and which to buy to obtain maximum profit margin.

(v) In selecting equipment and evaluating methods of improvement that maximize profit margin.

(vi) In planning most profitable match of sales requirements to plant capacity that obtains a fair share of the market.

(vii) In design of optimal purchasing policies.

Formulation of Linear Programming Problem

The formulation of linear programming problem as a mathematical model involves the following basic steps.

Step 1: Find the key-decision to be made from the study of the solution. (In this connection, looking for variables helps considerably).

Step 2: Identify the variables and assume symbols x_1, x_2 …… for variable quantities noticed in step 1.

Step 3: Express the possible alternatives mathematically in terms of variables. The set of feasible alternatives generally in the given situation is:

$$[(x_1, x_2); x_1 > 0, x_2 > 0]$$

Step 4: Mention the objective quantitatively and express it as a linear function of variables.

Step 5: Express the constraints also as linear equalities/inequalities in terms of variables.

Some Definitions

(a) Solution:

Values of decision variables x_j (j = 1, 2 ….., n) which satisfy the constraints of a general L.P.P., is called the Solution to that L.P.P.

(b) Feasible Solution:

Any solution that satisfies all the constraints (including the non-negative ones) of the general L.P.P., is called a Feasible Solution.

(c) Basic Solution:

For a set of m simultaneous equations in n unknowns *(n>m)*, a solution obtained by setting *(n-m)* of the variables equal to zero and solving the remaining m equations in m unknowns is called a Basic Solution. The (n – m) number of variables which are set equal to zero are called Non Basic Variables and remaining m are called Basic Variables and constitute a basic solution.

(d) Basic Feasible Solution:

A feasible solution to a general L.P. problem which is also basic solution is called a Basic Feasible Solution.

(e) Optimal Feasible Solution:

Any basic feasible solution which optimize (maximize or minimize) the objective function of a general

L.P.P. is called an Optimal Feasible Solution to that L.P. problem.

(f) Degenerate Solution:

A basic solution to the system of equations is called Degenerate if one or more of the basic variables become equal to zero.

Illustration 1

A shopkeeper deals in two items – Wall hangings and Artificial plants. He has ₹ 50000 to invest and a space to store 100 pieces at the most. Costs of Wall hangings and Artificial plants are respectively ₹ 450 and ₹ 200 each. He can sell a Wall hanging at a profit of ₹ 80 and an Artificial plant at a profit of ₹ 37. Assuming that he can sell all the items that he buys, formulate a Linear Programming problem in order to maximize his profit.

Solution:

Let x_1 be the number of Wall hangings and x_2 be the number of Artificial plants that the dealer buys and sells. So the Decision Variables in this case are x_1 and x_2 respectively.

Total Profit of the shopkeeper can be expressed as $Z = 80x_1 + 37x_2$ and this is the Objective Function.

Similarly Total Cost of buying the items can be given as $(450x_1 + 200x_2)$ which should not exceed the investment capacity of the shopkeeper i.e ₹ 50000. It can be expressed as $450x_1 + 200x_2 \leq 50000$

Again there is a space limitation and the shopkeeper can store at the most 100 pieces. We can express this as

$x_1 + x_2 \leq 100$

Moreover x_1 and x_2, being physical quantities, should be non-negative. So $x_1 \geq 0$ and $x_2 \geq 0$

Thus the mathematical formulation of the given LPP is –

Maximize $Z = 80x_1 + 37x_2$

Subject to the constraints

$450x_1 + 200x_2 \leq 50000$

$x_1 + x_2 \leq 100$

$x_1 \geq 0$ and $x_2 \geq 0$

Illustration 2

A dealer of cement has two warehouses M and N with stocks of 30000 and 20000 bags of cement respectively. Three customers A, B and C have placed order on the dealer for 15000, 20000 and 15000 bags respectively. Costs of transportation per 1000 bags of cement from different warehouses to different customers are given below.

	Transportation Cost (₹ 000) per 1000 bags		
To	A	B	C
From			
M	40	20	20
N	20	60	40

The dealer wants to find how to fulfill the orders so that the transportation cost is minimum. Formulate the problem.

Solution:

As transportation costs are given per 1000 bags, we assume 1 unit = 1000 bags

Let Warehouse M supplies x_1 units to A and x_2 units to B. As the stock of M is 30000 bags or 30 units, so C gets $(30 - x_1 - x_2)$ units from M.

Total requirement of A is 15000 bags or 15 units. Of this x_1 is supplied from M. Thus remaining $(15 - x_1)$ units is to be supplied from N.

Similarly B gets $(20 - x_2)$ units from N and C gets $[15 - (30 - x_1 - x_2)] = x_1 + x_2 - 15$ units from N.

Using the supplied values of Transportation Cost per unit we express Total Transportation Cost as -

$Z = 4000x_1 + 2000x_2 + 2000(30 - x_1 - x_2) + 2000(15 - x_1) + 6000(20 - x_2) + 4000(x_1 + x_2 - 15)$

Or, $Z = 4000x_1 - 2000x_2 + 150000$

As the problem deals with units of cement bags, each of the units mentioned above should be non-negative. Hence the constraints are –

$x_1 \geq 0$, $x_2 \geq 0$, $30 - x_1 - x_2 \geq 0$ Or, $x_1 + x_2 \leq 30$, $15 - x_1 \geq 0$ Or, $x_1 \leq 15$, $20 - x_2 \geq 0$ Or, $x_2 \leq 20$

$x_1 + x_2 - 15 \geq 0$ Or, $x_1 + x_2 \geq 15$

Thus the mathematical formulation of the given LPP is –

Minimize $Z = 4000x_1 - 2000x_2 + 150000$

Subject to the constraints

$x_1 + x_2 \leq 30$

$x_1 + x_2 \geq 15$

$x_1 \leq 15$

$x_2 \leq 20$

$x_1 \geq 0, x_2 \geq 0$

Methods of Solving Linear Programming Problems

Following two methods of finding an optimal solution of a Linear Programming Problem are mostly used.

(1) Graphical Method

(2) Simplex Method

1. Graphical Method

Graphical Method is generally used for solving Linear Programming Problems having two or three variables. In fact for problems with three variables also the application of this method is rather rare. Due to this limitation of handling only a few variables, this method is not applied for solving industrial problems which normally contains more variables.

Steps in solving a LPP graphically are as follows-

(1) Formulate the problem to get a linear Objective Function which is subjected to a number of Constraints.

These Constraints are generally in the form of linear inequality. Sometimes they may be in the form of linear equations also.

(2) Draw the graphs for each of the Constraints. For inequalities the graphs are represented by regions or space and for equations they are represented by straight lines. For "Greater than" or "Greater than equal to" type inequalities the region will be above the constraint line. For " Less than" or "Less than equal to" type inequalities it will be below the constraint line.

(3) Identify the feasible region or solution space. This is actually the area or the space which satisfies all the inequalities simultaneously.

(4) Select any one of the following two methods to get the ultimate solution.

- Corner Point Method necessitates identification of each corner point or vertex of the Feasible Region and subsequently find their coordinates. This can either be done directly from the graph paper or be done by solving the simultaneous equations of the constraint lines which form a particular corner. Thereafter the value of the Objective Function is to be computed at each of the corner points mentioned above by substituting the values of the coordinates. For a problem of Maximization take that highest value of the Objective Function and for a Minimization problem, take the lowest value of the Objective Function. This value along with the coordinates of the point giving the value, will be the solution.
- Iso-profit or Iso-cost Method deals with choosing any arbitrary numerical value of the Objective Function such that the corresponding straight line falls within the feasible region. Now move the line parallel to itself over the feasible region so that it passes through all the corner points. For maximization, the corner corresponding to Iso-profit line farthest from the origin gives maximum value. For minimization, the corner corresponding to Iso-cost line closest to the origin gives minimum value.

Illustration 3 (Maximization problem solved by Corner Point Method)

One kind of cake requires 200 grams of flour and 25 grams of fat and another kind of cake requires 100 grams of flour and 50 grams of fat. Find the maximum number of cakes that can be made from 5 kgs. of flour and 1 kg. of fat assuming there is no shortage of other ingredients required for making cakes. Formulate LPP based on the information given and solve graphically.

Solution:

Let x and y be the number of cakes of the first and second kinds respectively. Thus total number of cakes can be expressed as $Z = x + y$

Limitations on the availability of flour is expressed as $200x + 100y \leq 5000$ Or, $2x + y \leq 50$

Also limitations on the availability of fat is expressed as $25x + 50y \leq 1000$ Or, $x + 2y \leq 40$

As both x and y are physical quantities, they have to be non-negative i.e $x \geq 0$ and $y \geq 0$

So the mathematical formulation of the given LPP is –

Maximize $Z = x + y$

Subject to the constraints

$2x + y \leq 50$, $x + 2y \leq 40$, $x \geq 0$ and $y \geq 0$

For solving the problem graphically first of all we need to get the Feasible Region where all the above and the space constraints are satisfied simultaneously. To get that we have to plot the constraint lines as explained below.

Converting the first constraint into an equation we get 2x + y = 50 Or, 2x/50 + y/50 = 1 Or, x/25 + y/50 = 1

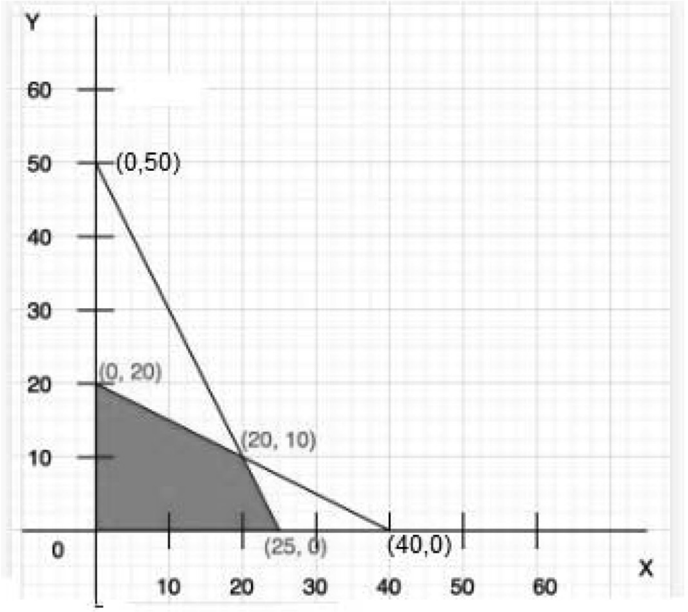
This is a straight line passing through the points (25,0) and (0,50). These points are plotted on a graph and subsequently joined to get the constraint line. The space below this line represents the constraint $2x + y \leq 50$

Similarly the second constraint is converted into the equation x + 2y = 40 Or, x/40 + y/20 = 1

This is also a straight line passing through the points (40.0) and (0.20). Points are plotted on the same graph and subsequently joined to get the line for the second constraint. The space below this line represents the constraint $x + 2y \leq 40$.

The lines for the other two constraints $x \geq 0$ and $y \geq 0$ i.e x = 0 and y = 0 are the two axes and the space above x axis denotes the region of $y \geq 0$ as well as the space to the right of y axis denotes the region of $x \geq 0$

The region common to all the above mentioned spaces denotes the Feasible Region. It is shown as the shaded area in the graph below.



Coordinates of the corner points of the Feasible Region are marked in the diagram above.

[It can be mentioned that the points (25,0) and (0,20) are already obtained and plotted to draw the constraint lines. But the coordinates of the point (20,10) have to be obtained either directly from the graph paper or by solving the simultaneous equations of the constraint lines i.e 2x + y = 50 and x + 2y = 40. Solution of simultaneous equations is a preferred option because for the problems where the coordinates are fraction, the graph will not give a perfect value.]

Now the value of the Objective Function is computed at each of the Corner Points and shown in the table below.

Coordinates of the Corner Point	Value of the Objective Function (Z = x + y)
(25, 0)	25 + 0 = 25
(20, 10)	20 + 10 = 30
(0, 20)	0 + 20 = 20

It is clear from the above values that Z is maximum at the point (20,10)

Thus the solution of the LPP is given as Z_{max} = 30 and the corresponding values of the Decision Variables are x_1 = 20 and x_2 = 10. In other words, the maximum number of cakes to be made is 30 and this is achieved by making 20 nos. of the first type and 10 nos. of the second type.

Illustration 4 (Minimization problem solved by Corner Point Method)

Mr. Lal is on a low cholesterol diet. During lunch at the office canteen he always chooses between two particular types of meal – Type A and Type B. The table below lists the amount of protein, carbohydrates and vitamins each meal provides along with the amount of cholesterol (which he is trying to minimize). He needs at least 200 grams of protein, 960 grams of carbohydrates and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have Type A meal and how many days the Type B meal so that he gets adequate amount of protein, carbohydrates and vitamins and at the same time minimizes his cholesterol intake? Use Graphical Method.

	Type A meal	Type B meal
Protein (Grams)	8	16
Carbohydrates (Grams)	60	40
Vitamins (Grams)	2	2
Cholesterol (Miligrams)	60	50

Solution:

Let, x = No. of days Mr. Lal will take Type A meal & y = No. of days Mr. Lal will take Type B meal

Since the goal is to minimize Mr. Lal₹s cholesterol intake, the Objective Function should represent the total cholesterol provided by both the meals.

So the Objective Function is Z = 60x + 50y

The constraints are given as follows –

$8x + 16y \geq 200$ (Constraint associated with the total protein provided by the two types of meals)

Or, $x + 2y \geq 25$

$60x + 40y \geq 960$ (Constraint associated with the total carbohydrates provided by the two types of meals)

Or. $3x + 2y \geq 48$

$2x + 2y \geq 40$ (Constraint associated with the total vitamins provided by the two types of meals

Or, $x + y \geq 20$

Also x and y being number of days cannot be negative i.e $x \geq 0$ and $y \geq 0$

So the formulated LPP can be stated as –

Minimize $Z = 60x + 50y$

Subject to the constraints

$x + 2y \geq 25$

$3x + 2y \geq 48$

$x + y \geq 20$

$x \geq 0$ and $y \geq 0$

To find the feasible region, first of all the straight lines corresponding to the above constraints are drawn using the method followed in the previous illustration.

$x + 2y = 25$ Or, $x/25 + y/12.5 = 1$ is the first constraint line and it passes through (25,0) and (0,12.5)

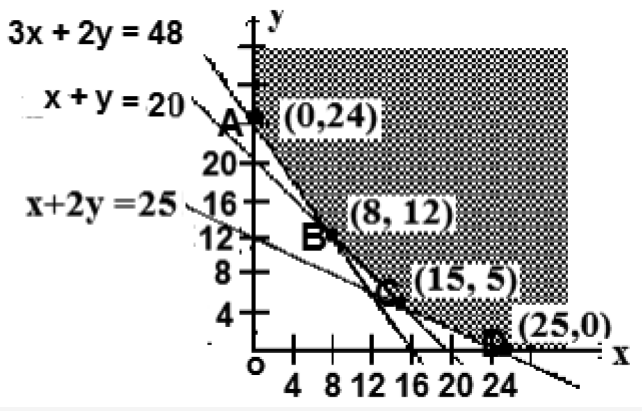
$3x + 2y = 48$ Or, $x/16 + y/24 = 1$ is the second constraint line which passes through (16,0) and (0,24)

$x + y = 20$ Or, $x/20 + y/20 = 1$ is the third constraint line and it passes through (20,0) and (0,20)

$x = 0$ is the axis of y and $y = 0$ is the axis of x

Now the constraint inequalities are graphed and the common region of the same is shaded as shown in the diagram below. It can be mentioned that the region of feasibility in this case is unbounded on the upper side. But that is not a matter of concern because the problem deals with minimization of the Objective Function which is confined to the corner points of the lower boundary of the envelope. As per the diagram, such corner points are A, B, C and D. Of these points coordinates of A and D are directly available from the graph because they lie on the axes. Coordinates of C and D can also be obtained from the graph. But it is suggested to get those by solving the simultaneous equations.

Coordinates of B are obtained by solving the equations $3x + 2y = 48$ and $x + y = 20$ and those of C are obtained by solving $x + y = 20$ and $x + 2y = 25$



Now the value of the Objective Function is computed at each of the corner points and shown in the table below.

Coordinates of the Corner Point	Value of the Objective Function (Z = 60x+50y)
A (0,24)	(60×0)+(50×24) = 1200
B (8,12)	(60×8)+(50×12) = 1080
C (15,5)	(60×15)+(50×5) = 1150
D (25,0)	(60×25)+(50×0) = 1500

It is clear from the table above that the value of the Objective Function is minimum at B (8, 12).

Thus the solution of the LPP is given as Z_{min} = 1080 miligrams and the corresponding values of the decision variables are x = 8 and y = 12

Hence Mr. Lal should take Type A meal for 8 days and Type B for 12 days to intake least cholesterol.

Illustration 5 (Maximization problem solved by Iso-profit Method)

Solve graphically the following LPP –

Maximize $M = 50x_1 + 60x_2$

subject to the constraints $2x_1 + x_2 \leq 300$, $3x_1 + 4x_2 \leq 509$, $4x_1 + 7x_2 \leq 812$, $x_1 \geq 0$, $x_2 \geq 0$

Solution:

Decision variables are x_1 and x_2 & the Objective function is $M = 50x_1 + 60x_2$

1st Constraint is $2x_1 + x_2 \leq 300$. Corresponding line is $L_1 : 2x_1 + x_2 = 300$ Or, $x_1/150 + x_2/300 = 1$

2nd Constraint is $3x_1 + 4x_2 \leq 509$. Corresponding line is $L_2 : 3x_1 + 4x_2 = 509$ Or, $x_1/169.67 + x_2/127.25 = 1$

3rd Constraint is $4x_1 + 7x_2 \leq 812$. Corresponding line is $L_3 : 4x_1 + 7x_2 = 812$ Or, $x_1/203 + x_2/116 = 1$

4th Constraint is $x_1 \geq 0$. Corresponding line is $x_1 = 0$ & this the vertical axis

5th Constraint is $x_2 \geq 0$. Corresponding line is $x_2 = 0$ & this is the horizontal axis.

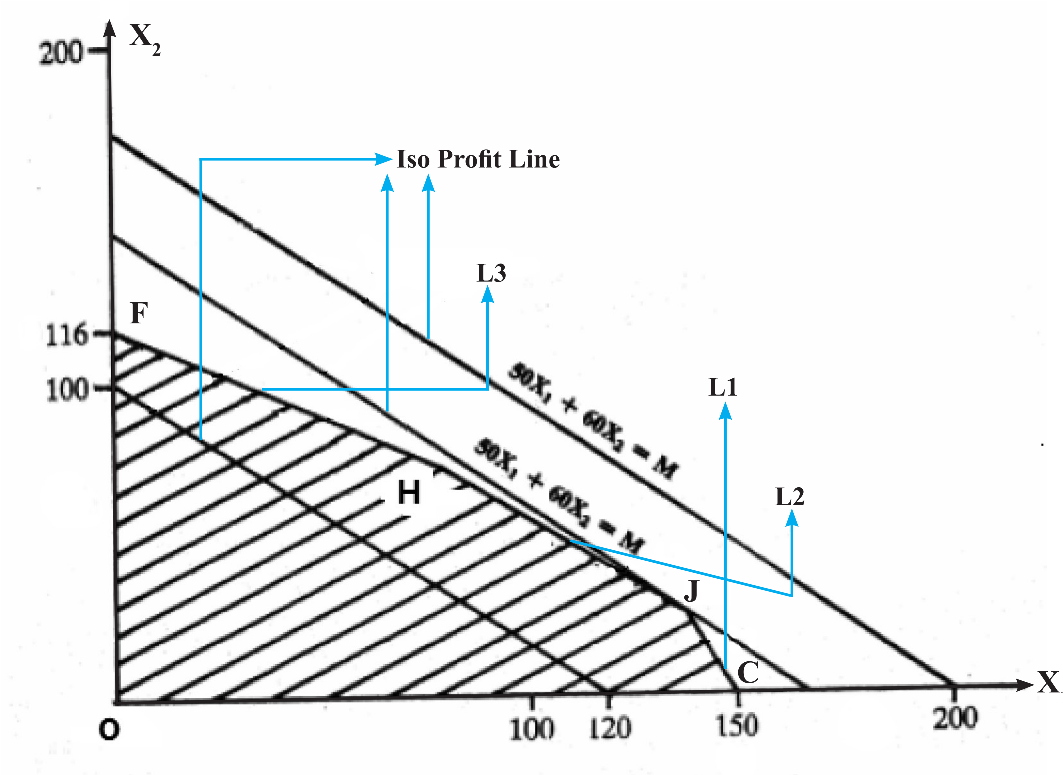
These constraints are graphed by following the method explained in the above illustrations. Subsequently the feasible region is obtained and shaded as shown in the figure below. This is a convex polygon having vertices at O, F, H, J and C. Coordinates of O, F & C are easily obtained from the graph and they are respectively (0,0), (0,116) & (150,0) Coordinates of H and J are obtained by solving the simultaneous equations. H is the point of intersection of L_2 and L_3 & J is the point of intersection of L_1 and L_2

For finding coordinates of J we have $2x_1 + x_2 = 300$ — (1) and $3x_1 + 4x_2 = 509$ — (2)

Substituting $x_2 = 300 - 2x_1$ from (1) into (2) we have $3x_1 + 4(300 - 2x_1) = 509$ Or, $5x_1 = 691$ Or, $x_1 = 138.2$

Substituting $x_1 = 138.2$ in (1) we have $2 \times 138.2 + x_2 = 300$ Or, $x_2 = 23.6$. So J = (138.2, 23.6)

Similarly coordinates of H can be calculated.



The objective function $M = 50x_1 + 60x_2$ represents a family of straight lines with slope = - 5/6. Some of these lines will intersect with the feasible region and contain many feasible solutions while the others will not intersect and contain no feasible solution. Our aim is to find out that line of the family which intersects with the feasible region and reach farthest out from the origin (as this is a problem of maximization. For minimization it is just the reverse) Farthest is the line from the origin, greater is the value of M.

From the diagram it is clear that J is one such point farthest from the origin and lying on $50x_1 + 60x_2 = M$. Also coordinates of J are (138.2, 23.6).

At this point the value of the Objective Function is $50 \times 138.2 + 60 \times 23.6 = 8326$ and this is the required maximum value.

So the solution to the LPP is $M_{max} = 8326$ and $x_1 = 138.2$ & $x_2 = 23.6$

[Note – In the above illustration the solution is unique because there is only one point J on the feasible region through which the Iso-profit line is passing. But there can be situations where the Iso-profit (or Iso-cost, as the case may be) line or the Objective Function line has the same slope as one of the constraint lines. In that case the Iso-profit (or the Iso-cost) line will coincide with one of the outer boundary lines of the Feasible Region and infinite number of optimal solutions will result. This is a special case of LPP called **Multiple Optima** situation.]

2. Simplex Method

Simplex Method is applicable to any LPP. There is no theoretical restriction on the number of Decision Variables or Constraints present in a particular LPP. The computational procedure of this This method was developed in mid 1947 by George Dantzig while working on planning methods of U.S Army Air Force to have a better utilization of the scarce resources during World War II. Computational procedure of the method is based on the property that Optimum solution to any LPP, if it exists, always occurs at one of the corner points of the feasible solution space. It is a systematic and efficient procedure for finding corner point solution and proceeding to attain optimality. As origin is one of the corner points of the solution space, evaluation always starts from this. This solution is called Initial Basic Feasible solution. This is then tested to find out if further improvement in the value of the Objective Function is possible by moving to the adjacent corner point of the feasible solution space. This iterative search is continued till an optimal solution is reached.

1. **Basic terms used in Simplex Method are as follows –**

- **Standard Form** of LPP refers to the form in which all the constraints are written as equalities. The optimum solution of the standard form of LPP is same as the optimum solution of the original formulation.
- **Slack Variable** is the variable used to convert a less than or less than equal to type constraint inequality into an equation. Contribution of this variable towards the Objective Function is zero. The value of this variable indicates the quantity of unused resources. For a constraint $x_1 + x_2 + x_3 \leq 20$ the form will change to $x_1 + x_2 + x_3 + s_1 = 20$ after introduction of the Slack Variable s_1
- **Surplus Variable** is the variable used to convert a more than or more than equal to type constraint inequality into an equation. Contribution of this variable towards the Objective Function is zero. It is interpreted as the quantity over and above the required minimum level. For a constraint $3x_1 + 2x_2 \geq 55$ the form will change to $3x_1 + 2x_2 - s_2 + A_1 = 55$ after introduction of the Surplus Variable s_2
- **Artificial Variable** – These variables are fictitious and are introduced only for computational purposes. These are used along with the surplus variables (shown as A1 above) for more than or more than equal to type constraints. For equal to type constraints they are introduced alone. Their contribution towards Objective Function is to cause a very high penalty (–M for a maximization problem and +M for a minimization problem)
- **Simplex Table** is used to keep track of the calculations made at each iteration.
- **Product Mix** is a column in the simplex table that contains all the variables in the solution
- **Basis** is the set of variables which are not equal to zero in the current basic solution and are listed in the product mix column. The variables which make up the Basis are termed as Basic Variables and the remaining variables are called Non basic Variables.
- **Z_j Row** is the row containing the figures for gross profit or loss given up by adding one unit of a variable into the solution.
- **Net Evaluation or Index Row** contains the net profit or loss that results by introducing one unit of the variable indicated in that column in the solution. Numerical figures in Index Row are also called Shadow Prices or Accounting Prices. Thus a positive (or negative) figure in the Index Row indicates an algebraic increment (or reduction) in the Objective Function if one unit of the variable at the head of that column is introduced in the Basis. Index Row is also called $(C_j - Z_j)$ Row.
- **Pivot or Key Column** is the one with largest positive number in the Index Row for a maximization problem or the largest negative number for a minimization problem. It indicates which variable will enter in the next step of solution.

- **Pivot or Key Row** corresponds to the variable that will depart from the Basis to accommodate the new variable to be entered as per the Key Column determined earlier. The departing variable will correspond to the minimum positive ratio found by dividing the Quantity Column values by the Key Column values for each row.
- **Pivot or Key Element** is the element of Simplex Table at the junction of the Key Row and Key Column.

2. Steps to be followed for solution of LPP (Maximization case) by Simplex Method

(1) Formulate the problem mathematically.

(2) Express the mathematically formulated problem in the Standard Form by introducing Slack variables to convert the constraint inequalities into equalities.

(3) Set up the Initial Basic Feasible Solution by assigning zero value to all the decision variables and represent it in the initial Simplex Table. Complete the Table by adding the two final rows for C_j and ($C_j - Z_j$) or the Index Row. Here Cj = Contribution per unit and these are nothing but the coefficients of the variables in the Objective Function in the Standard Form. Also $Z_j = S(C_{Bj}.a_{ij})$ where C_{Bj} = Coefficients of the current basic variables in the Objective function and a_{ij} = Values of the elements in the matrix

(4) Examine if all the elements in the Index Row are negative. If that is so then the current solution is optimum. In case there exist some positive value then the current solution is not optimal and scope of further improvement is there. For this one Basic Variable has to be removed from the Basis and to be replaced by a Non Basic variable.

(5) Determine which variable to enter into the Solution mix next. For this, identify the column in which the numerical figure in the Index Row is highest positive. This gives the Pivot Column and the variable corresponding to this column will enter into the next table.

(6) Determine which variable to depart from the Solution mix next. For this divide each number in the Quantity Column by the corresponding number in the Pivot Column to get the respective ratios. Identify the row having minimum of these ratios. Consider only the positive ratios here. This Row is the Pivot Row. The variable corresponding to the Pivot Row will depart from the Basis to accommodate for the entering variable obtained in step (5) at its place.

(7) Identify the Key element which is at the junction of the Pivot Row and Pivot Column.

(8) Compute new values for the Pivot Row by simply dividing each number in the Pivot Row by the Key element.

(9) Compute the new values for each of the other rows by using the following formula -

New row number = (Old row No.) – [(Corresponding No. in the Pivot Row) × (Corresponding Fixed Ratio)]

where, Corresponding Fixed Ratio = Old No. in the Pivot Column / Key No.

(10) Make new entries in the C_B column and X_B column in the new table of the current solution.

(11) Compute Z_j and ($C_j - Z_j$) as explained in step (3) above. If all the elements in the Index Row or ($C_j - Z_j$) row are zero or negative then an optimal solution has been achieved.

(12) If any value in the Index Row is positive then repeat the steps (5) through (11) given above.

[**Note:** Nomenclature of the different columns of the Simplex Table is provided in Table 1 of Illustration 6 below. Besides, the method of calculation of the values of CB and Z_j are also explained in the same Table.]

Special Cases in Linear Programming

(1) Infeasible Solution: Situation of infeasibility occurs when there is no solution to an LPP which satisfies all the constraints simultaneously. Graphically it means that no feasible region exists and the same can occur if the problem is formulated with conflicting constraints.

Simplex Method gives a clear indication that no feasible solution is possible. When at least one of the Artificial variables remains present as the Basic variable in the final solution table then it is considered as a situation of Infeasible solution.

(2) Unbounded Solution: It means the feasible region is not bounded in any respect. For LPP with constraints such that the feasible region extends to infinity on the upper right side of the graph it is said that there is unbounded solution to the problem.

In case of Simplex Method, the situation of unbounded solution occurs if all the ratios in the Minimum Ratio column (which is the last column of any Simplex table) are either negative or infinite.

(3) Multiple Optimum Solution: Graphically this case is encountered when the slope of Iso-profit or Iso-cost lines of the Objective Function matches that of any one of the Constraint lines. In other words the Iso-profit or Iso-cost lines will be parallel to any one of the constraint lines. In fact there will be infinite number of optimum solutions.

This situation can be recognized in the Simplex Method when one of the non-basic variables in the (Cj – Zj) row will have a zero value in the final solution table. To get the other solution of the problem one has to bring in the non-basic variable into the Basis.

Illustration 6

A firm manufactures and sells two products Alpha and Beta. Each unit of Alpha requires 1 hour of machining and 2 hours of skilled labour, whereas each unit of Beta uses 2 hours of machining and 1 hour of labour. For the coming month the machine capacity is limited to 720 machine hours and the skilled labour is limited to 780 hours. Not more than 320 units of Alpha can be sold in the market during a month.

(i) Develop a suitable model that will enable determination of the optimal product mix.

(ii) Determine the optimal product-mix and the maximum contribution if Unit contribution from Alpha is ₹ 6 and from Beta is ₹ 4.

(iii) What will be the incremental contribution per unit of the machine hour, per unit of labour, per unit of Alpha saleable?

Solution:

(a) Given information is summarized in the table below

Products	Machining	Skilled Labour	Contribution
Alpha	1 hr	2 hr	6/-
Beta	2 hr	1 hr	4/-
Available hours	720 hr	780 hr	

Let x_1 be the no. of units of Alpha produced and x_2 be the no. of units of Beta produced.

Objective function: is -

Maximize. $Z = 6x_1 + 4x_2$.

Subject to the constraints

$x_1 + 2x_2 \leq 720$

$2x_1 + x_2 \leq 780$

$x_1 \leq 320$

and $x_1, x_2 \geq 0$

(b) Introducing non negative Slack Variables S_1, S_2 and S_3 to convert the less than equal to type constraints into equations we can rewrite the constraints as below.

$x_1 + 2x_2 + S_1 = 720$

$2x_1 + x_2 + S_2 = 780$

$x_1 + S_3 = 320$

Thus the given LPP in Standard Form should be -

Maximize, $Z = 6x_1 + 4x_2 + 0.S_1 + 0.S_2 + 0.S_3$

[Contribution of Slack Variables to the Objective Function is zero because they represent unused resources]

Subject to the constraints

$x_1 + 2x_2 + S_1 = 720$

$2x_1 + x_2 + S_2 = 780$

$x_1 + S_3 = 320$

$x_1, x_2, S_1, S_2 \& S_3 \geq 0$

Now setting $x_1 = x_2 = 0$ we get an Initial Solution as $S_1 = 720$, $S_2 = 780$ and $S_3 = 320$ & the corresponding value of Z = 0. The results are summarized in the Initial Simplex table given below.

Table – 1: Initial Simplex Table

Profit per unit of Basic variables Column	**Basic variables Column**	**Values of Basic Variables Column**	**Decision Variables Column**		**Slack Variables Column**			**Ratio $[X_B / (a_{ij})_{Key\ Col}]$ Column**	
	Contribution per unit (C_j) Row		**6**	**4**	**0**	**0**	**0**		
C_B	Basic variables (B)	Solution Values ($b = X_B$)	x_1	x_2	S_1	S_2	S_3		
0	S_1	720	$1 = a_{11}$	$2 = a_{12}$	$1 = a_{13}$	$0 = a_{14}$	$0 = a_{15}$	720/1 = 720	
0	S_2	780	$2 = a_{21}$	$1 = a_{22}$	$0 = a_{23}$	$1 = a_{24}$	$0 = a_{25}$	780/2 = 390	
0	S_3	320	$^*1 = a_{31}$	$0 = a_{32}$	$0 = a_{33}$	$0 = a_{34}$	$1 = a_{35}$	320/1 = 320	Key Row
$Z_B = \sum(C_B.X_B) = 0$		$Z_j = \sum(C_B.a_{ij})$	0	0	0	0	0	S_3 departs & x_1 enters in the next table	
		$(C_j - Z_j)$	6	4	0	0	0		
			Key Col						

From the Initial Simplex Table it is seen that the Key or Pivot Column is the 1st Column of the matrix because the value of $(C_j - Z_j)$ is maximum against it. Also the Key or Pivot Row is the 3rd Row of the matrix because the Ratio is minimum against it. So Key element is $a_{31} = 1$. Hence S_3 departs and x_1 enters into the next Table of Simplex.

Table – 2: Second Simplex Table (Improved Solution)

	Contribution per unit (C_j)		6	4	0	0	0	Ratio	
C_B	Basic variables (B)	Solution Values ($b = X_B$)	x_1	x_2	S_1	S_2	S_3	$[X_B / (a_{ij})_{Key\ Col}]$	
0	S_1	400	$0 = a_{11}$	$2 = a_{12}$	$1 = a_{13}$	$0 = a_{14}$	$-1 = a_{15}$	400/2 = 200	
0	S_2	140	$0 = a_{21}$	$^*1 = a_{22}$	$0 = a_{23}$	$1 = a_{24}$	$-2 = a_{25}$	140/1 = 140	Key Row
6	x_1	320	$1 = a_{31}$	$0 = a_{32}$	$0 = a_{33}$	$0 = a_{34}$	$1 = a_{35}$	$320/0 = \infty$	
$Z_B = \sum(C_B.X_B) = 1920$		$Z_j = \sum(C_B.a_{ij})$	6	0	0	0	6	S_2 departs & x_2 enters in the next table	
		$(C_j - Z_j)$	0	4	0	0	- 6		
				Key Col					

The second Simplex Table is achieved by following the steps given below.

(1) The elements of the row corresponding to the Pivot Row of Table 1 is obtained by dividing each element by the Key element of the previous table. As Key element is 1, all the elements will remain same as the previous table i.e 3rd row remains same.

(2) Other rows are filled up by using the following formula

New row number = (Old row No.) – [(Corresponding No. in the Pivot Row) × (Corresponding Fixed Ratio)]

where, Corresponding Fixed Ratio = Old No. in the Pivot Column / Key No.

For the 2nd Row:- Old number in the Pivot Column = 2 & Key No. = 1, So Fixed Ratio = 2/1 = 2

New (a_{21}) = Old (a_{21}) – [Old (a_{31}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{31})]

$= 2 - [1 \times 2] = 0$

New (a_{22}) = Old (a_{22}) – [Old (a_{32}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{32})]

$= 1 - [0 \times 2] = 1$

New (a_{23}) = Old (a_{23}) – [Old (a_{33}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{33})]

$= 0 - [0 \times 2] = 0$

New (a_{24}) = Old (a_{24}) – [Old (a_{34}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{34})]

$= 1 - [0 \times 2] = 1$

New (a_{25}) = Old (a_{25}) – [Old (a_{35}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{35})]

$= 0 - [1 \times 2] = -2$

New Solution value for 2nd row = Old Solution value for 2nd row – [(Solution value in the Pivot row) × Fixed Ratio]

= 780 – [320 × 2] = 780 – 640 = 140

For the 1st Row:- Old number in the Pivot Column = 1 & Key No.= 1, So Fixed Ratio = 1/1 = 1

New (a_{11}) = Old (a_{11}) – [Old (a_{31}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{31})]

= 1 – [1×1] = 0

New (a_{12}) = Old (a_{12}) – [Old (a_{32}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{32})]

= 2 – [0×1] = 2

New (a_{13}) = Old (a_{13}) – [Old (a_{33}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{33})]

= 1 – [0×1] = 1

New (a_{14}) = Old (a_{14}) – [Old (a_{34}) x Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{34})]

= 0 – [0×1] = 0

New (a_{15}) = Old (a_{15}) – [Old (a_{35}) × Fixed Ratio] [Here corresponding number in the Pivot Row = Old (a_{35})]

= 0 – [1×1] = –1

New Solution value for 1st row = Old Solution value for 1st row – [(Solution value in the Pivot row) × Fixed Ratio]

= 720 – [320 × 1] = 720 – 320 = 400

From the Second Simplex Table it is seen that the Key or Pivot Column is the 2nd Column of the matrix because $(C_j – Z_j)$ is maximum against it. Also the Key or Pivot Row is the 2nd Row of the matrix because the Ratio is minimum against it. So Key element is a_{22} = 1. Hence S_2 departs and x_2 enters into the next Table of Simplex.

Table – 3: Third Simplex Table (Improved Solution)

		Contribution per unit (C_j)	6	4	0	0	0		
C_B	Basic variables (B)	Solution Values $(b = X_B)$	x_1	x_2	S_1	S_2	S_3	Ratio $[X_B / (a_{ij})_{Key\ Col}]$	
0	S_1	120	$0 = a_{11}$	$0 = a_{12}$	$1 = a_{13}$	$-2 = a_{14}$	$*3 = a_{15}$	120/3 = 40	Key Row
4	x_2	140	$0 = a_{21}$	$*1= a_{22}$	$0 = a_{23}$	$1 = a_{24}$	$-2 = a_{25}$	140/(-2) = -70	
6	x_1	320	$1 = a_{31}$	$0 = a_{33}$	$0 = a_{33}$	$0 = a_{34}$	$1 = a_{35}$	320/1 = 320	
$Z_B = S(C_BX_B)$ = 2480		$Z_j = S(C_Ba_{ij})$	6	4	0	4	- 2	S_1 departs & S_3 enters in the next table	
		$(C_j – Z_j)$	0	0	0	- 4	2		
							Key Col		

The solution is non-optimal because there exists a positive entry in the Index Row. Hence Table 4 is constructed and given below.

Table – 4: Fourth Simplex Table (Optimal Solution)

		Contribution per unit (C_j)	6	4	0	0	0
C_B	Basic variables (B)	Solution Values ($b = X_B$)	x_1	x_2	S_1	S_2	S_3
0	S_3	40	$0 = a_{11}$	$0 = a_{12}$	$\frac{1}{3} = a_{13}$	$-\frac{2}{3} = a_{14}$	$1 = a_{15}$
4	x_2	220	$0 = a_{21}$	$^*1 = a_{22}$	$\frac{2}{3} = a_{23}$	$-\frac{1}{3} = a_{24}$	$0 = a_{25}$
6	x_1	280	$1 = a_{31}$	$0 = a_{32}$	$-\frac{1}{3} = a_{33}$	$\frac{2}{3} = a_{34}$	$0 = a_{35}$
$Z_B = \Sigma(C_B X_B)$ =2560		$Z_j = \Sigma(C_B a_{ij})$	6	4	2/3	8/3	0
		$(C_j - Z_j)$	0	0	- 2/3	- 8/3	0

As the entries in $(C_j – Z_j)$ Row are either zero or negative, the solution is Optimal.

Maximum value of Z = 2560 which corresponds to $x_1 = 280$ and $x_2 = 220$

Thus the optimum product mix is 280 units of Alpha and 220 units of Beta which will result in maximum contribution of ₹ 2560

(c) From the final table of Simplex it can be said that the non-basic variables are S_1 and S_2 which are the Slack Variables used corresponding to the constraints of Machining hours and Skilled Labour hours. Entry against these variables in the Z_j row of the final table of Simplex indicates Shadow Cost figures against these. In other words, these represent the decrease in the optimum value of the Objective Function resulting from a unit increase in them.

Incremental contribution per unit of Machining hour = Entry in the $(C_j – Z_j)$ row of the final Simplex table against S_1 = ₹ 2/3 (loss)

Incremental contribution per unit of Labour hour = Entry in the $(C_j – Z_j)$ row of the final Simplex table against S_2 = ₹ 8/3 (loss)

Incremental contribution per unit of Alpha saleable = Entry in the $(C_j – Z_j)$ row of the final Simplex table against $S_3 = 0$

Each of the above mentioned contribution figures will reduce the optimum value of the Objective Function.

Illustration 7

Sri Lanka, the third largest tea producing country has a production share of 9% of the international market and one of the world₹s leading exporters with a share of 19% of the global demand. Thus tea industry is crucial to enhance their economic competitiveness in the world market. The nature of the highly competitive global market has made scientific and reasonable production management increasingly important for tea companies to differentiate themselves from competitors. In order to enhance their competitive position, Sri Lankan tea manufacturers are giving serious thought to use optimization techniques like Linear Programming to find their best product mix to achieve maximization of profit. Dulwan Tea Company, established in 1974 is one of the leading tea exporters of the country. They use their own leaves which grow in their tea plantations. More than 2500 varieties of flavored and non-flavored tea products are produced and globally exported by the company. This brand is available in more than 90 countries in the world including UK, Poland, Canada, South Africa, Australia and New Zealand. Therefore how to optimize the production process yielding maximum profit is a critical and challenging task in front of the decision makers of Dulwan. After lot of deliberations among themselves, the management of Dulwan has decided to hire a Cost and Management consultant.

Accordingly they hired Mr. Kuppuswamy, a resident of Jafna, Sri Lanka and a well-known consultant of the island. In his first visit to the company the management explained to him the requirements and Mr. Kuppuswamy technically phrased the objective of the work as follows.

- To formulate a mathematical model that would suggest a viable product mix to ensure maximum profit of the company as well as evaluating performance of the proposed product mix.
- To highlight the peculiarities of using linear programming technique at a single operating procedure and prove that despite the obstacles, the application of the technique in determining the product mix enables Dulwan Tea Company to be more profitable than the otherwise.

Thereafter a team is formed from the existing employees of the company and under the guidance of Mr. Kuppuswamy they started working to formulate the problem as a Linear Programming model. Since the company is dealing with huge varieties of tea product, everybody could realize that solving such LPP manually is impossible. So it is decided to purchase a suitable software for the purpose and Mr. Kuppuswamy is requested to get at least three quotes from renowned global software companies. When the process is on, all of a sudden new opportunities open and the company decided to bid for supplying few of its very premium quality tea to the European market. But the management was not very sure as to which quality of tea they should try to sell so that the objective of profit maximization is fulfilled. Once again Mr. Kuppuswamy was approached and this time he decided to find the best product mix by solving the problem manually (as variety of very premium quality tea was not much and also the decision regarding which software to purchase not finalized).

During solution of the problem manually, at one stage the following Simplex Table is obtained

C_B	Product Mix	Quantity	x_1	x_2	x_3	s_1	s_2	s_3	A_1
2	x_1	4	1	2	1/2	0	0	1/4	0
0	s_2	12	0	0	–1	0	1	–1/2	0
0	s_1	12	0	6	0	1	0	1	–1
	C_j		2	4	1	0	0	0	– M
	Z_j	8	2	4	1	0	0	½	0
	$C_j – Z_j$		0	0	0	0	0	–1/2	–M

Answer the following questions, with proper explanation, related to the Simplex Table above.

i) How many varieties of very Premium quality tea are considered in the problem?

ii) Is the solution given in the Table above Optimal?

iii) What is the Objective Function?

iv) Is there any alternate solution to the problem?

v) Is the solution feasible?

vi) What is the optimum product mix and the maximum profit.

vii) If any alternate solution is possible then find it.

Solution:

i) From the table it is clear that there are three decision variables x_1, x_2 and x_3

So 3 varieties of very Premium quality tea are considered.

ii) The given table has either zero or negative entries in the Index Row or $(C_j – Z_j)$ Row. Also this is a problem of maximization. So the criteria of optimality is satisfied here. hence the solution is optimal

iii) C_j row represents the contribution per unit of different variables. Thus from the table, respective contributions of the Decision variables x_1, x_2 and x_3 are 2, 4 and 1. So the Objective Function is $Z = 2x_1 + 4x_2 + x_3$

iv) In the Optimal Table only decision variable x_1 is present in the Basis. Thus the other two decision variables x_2 and x_3 are Non-basic variables and corresponding to both the entries in the Index Row or $(C_j - Z_j)$ Row are zero. This indicates the presence of multiple optimum solutions. Hence there exists alternate solution to the problem.

v) As no Artificial Variable is present as Basic Variable in the final table, infeasibility of the solution is not applicable. Hence the solution is feasible.

vi) Optimum product mix is as follows –

Quantity of Type 1 quality premium tea to be produced = x_1 = 4 quantity units and no units for the other two varieties of tea and.Maximum profit = 8 money units

vii) From the answer given in (v) above, we can say that two alternate solutions are possible- either that will contain x_2 or x_3 So a new solution is find as below by arbitrarily choosing the column of x_2 as the Key Column.

Table -1 showing Optimal Solution

C_B	Product Mix	Quantity	x_1	x_2	x_3	s_1	s_2	s_3	A_1	Ratio	
2	x_1	4	1	2*	1/2	0	0	1/4	0	4/2 = 2	Key Row
0	s_2	12	0	0	–1	0	1	–1/2	0	12/0 = 0	
0	s_1	12	0	6	0	1	0	1	–1	12/6 = 2	
	C_j		2	4	1	0	0	0	–M		
	Z_j	8	2	4	1	0	0	½	0		
	$C_j – Z_j$		0	0	0	0	0	–1/2	–M		
				Key Column							

Minimum Ratio column has two tied ratios corresponding to two basic variables x_1 and s_1. We arbitrarily choose the row corresponding to x_1 as the Key Row. So the key element is 2 and the cell for that is shaded as shown. Hence in the next table x_2 enters and x_1 departs.

Further calculations are done by following the methodology explained in Illustration 6 before and shown in the next table below.

Table 2 showing Alternative Solution

C_B	Product Mix	Quantity	x_1	x_2	x_3	s_1	s_2	s_3	A_1	Ratio
4	x_2	2	1/2	1	1/4	0	0	1/8	0	
0	s_2	12	0	0	–1	0	1	–1/2	0	
0	s_1	0	–3	0	–3/2	1	0	1/4	–1	
	C_j		2	4	1	0	0	0	–M	
	Z_j	8	2	4	1	0	0	1/2	0	
	$C_j - Z_j$		0	0	0	0	0	–1/2	–M	

As all the entries in Index Row or $(C_j – Z_j)$ Row are zero or negative, the solution is optimal. Here also the maximum value of Objective Function is 8 money units and the optimum product mix is -

Quantity of Type 2 quality premium tea to be produced = x_2 = 2 quantity units and no units for the other two varieties

Illustration 8

A company possesses two manufacturing plants each of which can produce three products X, Y and Z from a common raw material. However, the proportions in which the products are produced are different in each plant and so are the plant's operating costs per hour. Data on production per hour costs are given below, together with current orders in hand for each product.

	Product			Operating cost per hour in ₹
	X	**Y**	**Z**	
Plant A	2	4	3	9
Plant B	4	3	2	10
Orders on hand	50	24	60	

You are required to use the simplex method to find the number of production hours needed to fulfill the orders on hand at minimum cost.

Interpret the main features of the final solution

Solution:

Let α be no. of hours of plant A in use

Let β be no. of hours of plant B in use

Objective function: Minimize $Z = 9\alpha + 10\beta$

Subject to the constraints:

$2\alpha + 4\beta \geq 50$ (Constraint for Orders in hand of Product X)

$4\alpha + 3\beta \geq 24$ (Constraint for Orders in hand of Product Y)

$3\alpha + 2\beta \geq 60$ (Constraint for Orders in hand of Product Z)

And $\alpha, \beta \geq 0$ (Non-negativity constraint)

Introducing non negative Surplus variables (S_1, S_2 & S_3) and Artificial variables (A_1 , A_2 & A_3) the constraints are rewritten as below

$2\alpha + 4\beta - S_1 + A_1 = 50$

$4\alpha + 3\beta - S_2 + A_2 = 24$

$3\alpha + 2\beta - S_3 + A_3 = 60$

So the LPP in Standard Form is given by –

Minimize $Z = 9\alpha + 10\beta + 0.S_1 + 0.S_2 + 0.S_3 + M.A_1 + M.A_2 + M.A_3$

[Contribution of Surplus variables towards Objective Function is zero always and that of Artificial variables is a very high value M for minimization problems and – M for maximization problems.]

Subject to the constraints –

$2\alpha + 4\beta - S_1 + A_1 = 50$

$4\alpha + 3\beta - S_2 + A_2 = 24$

$3\alpha + 2\beta - S_3 + A_3 = 60$

$\alpha, \beta, S_1, S_2, S_3, A_1, A_2$ & $A_3 \geq 0$

Setting $\alpha = \beta = S_1 = S_2 = S_3 = 0$ the Initial Basic Feasible Solution is given as $A_1 = 50$, $A_2 = 24$ & $A_3 = 60$ This is shown in the following Table 1.

Table 1: Showing Initial Solution

Contribution per unit (C_j)			9	10	0	0	0	M	M	M	Minimum Ratio	
C_B	Basic Variable	X_B	α	β	S_1	S_2	S_3	A_1	A_2	A_3		
M	A_1	50	2	4	–1	0	0	1	0	0	50/2 = 25	
M	A_2	24	4*	3	0	–1	0	0	1	0	24/4 = 6	Key Row
M	A_3	60	3	2	0	0	–1	0	0	1	60/3 = 20	
Z_B = 134M	–	Z_j	9M	9M	–M	–M	–M	M	M	M	α enters & A_2 departs	
		$(C_j - Z_j)$	9 - 9M	10-9M	M	M	M	0	0	0		
			Key Col.									

Now calculations are done by following the methodology explained before in Illustration 6 [with the exception in the method of determination of Key Column. This is a case of minimization and the Key Column is the one which is having highest negative entry in the Index Row that is $(C_j - Z_j)$ Row] and various improved solutions are obtained as given in the following tables of Simplex.

Table 2: Showing Improved Solution

Contribution per unit (C_j)			9	10	0	0	0	M	M	M	Minimum Ratio	
C_B	Basic Variable	X_B	α	β	S_1	S_2	S_3	A_1	A_2	A_3		
M	A_1	38	0	5/2	–1	1/2	0	1	–1/2	0	38/(5/2) =15.2	
9	α	6	1	*3/4	0	–1/4	0	0	1/4	0	6/(3/4) = 8	Key Row
M	A_3	42	0	–1/4	0	3/4	–1	0	–3/4	1	42/(-1/4)= -	
Z_B = 80M+54	–	Z_j	9	27/4 + 9M/4	–M	5M/4–9/4	–M	M	–5M/4+9/4	M	β enters & α departs	
	$(C_j - Z_j)$		0	13/4 – 9M/4	M	9/4–5M/4	M	0	9M/4–9/4	0		
				Key Col.								

[Note - Though calculations against the departing variable (A_2) column are shown here, but the same can be avoided because A_2 is an Artificial variable and any Artificial Variable once departed will not reappear. This is applicable for the following tables, too.]

Table 3: Showing Improved Solution

Contribution per unit (C_j)			9	10	0	0	0	M	M	M	Minimum Ratio	
C_B	Basic Variable	X_B	α	β	S_1	S_2	S_3	A_1	A_2	A_3		
M	A_1	18	–10/3	0	–1	*4/3	0	1	–4/3	0	18/(4/3) =13.5	Key Row
10	β	8	4/3	1	0	–1/3	0	0	1/3	0	8/(–1/3) = –24	
M	A_3	44	1/3	0	0	2/3	–1	0	–2/3	1	44/(2/3) = 66	
Z_B = 62M+80	–	Z_j	40/3 – 3M	10	–M	2M – 10/3	–M	M	–2M+10/3	M	S_2 enters & A_1 departs	
	$(C_j - Z_j)$		3M – 13/3	0	M	10/3 – 2M	M	0	3M – 10/3	0		
						Key Col.						

Table 4: Showing Improved Solution

Contribution per unit (C_j)			9	10	0	0	0	M	M	M	Minimum Ratio	
C_B	Basic Variable	X_B	α	β	S_1	S_2	S_3	A_1	A_2	A_3		
0	S_2	27/2	–5/2	0	–3/4	1	0	3/4	–1	0	–27/5	
10	β	25/2	1/2	1	–1/4	0	0	1/4	0	0	25	
M	A_3	40	*2	0	1/2	0	–1	–1/2	0	1	20	Key Row
Z_B = 40M+125	–	Z_j	5+2M	10	(M–5)/2	0	–M	(5–M)/2	0	M	α enters & A_3 departs	
		(C_j - Z_j)	4 –2M	0	(5–M)/2	0	M	(M–5)/2	M	0		
			Key Col.									

Table 5: Showing Optimal Solution

Contribution per unit (C_j)			9	10	0	0	0	M	M	M	Minimum Ratio
C_B	Basic Variable	X_B	α	β	S_1	S_2	S_3	A_1	A_2	A_3	
0	S_2	127/2	0	0	–1/8	1	–5/4	1/8	–1	5/4	
10	β	5/2	0	1	–3/8	0	1/4	3/8	0	–1/4	
9	α	20	1	0	1/4	0	–1/2	–1/4	0	1/2	
Z_B = 205	–	Z_j	9	10	–3/2	0	–2	3/2	0	2	
		(C_j - Z_j)	0	0	3/2	0	2	M – 3/2	M	M–2	

As all the values in the Index Row are either zero or positive, an Optimality condition is attained. Optimum solution is given as –

Minimum Operating Cost for meeting the production target of orders on hand = ₹ 205 and this can be achieved by operating plant A for α = 20 hours and Plant B for β = 5/2 hours.

Duality

Every LPP is associated with another mirror image problem based on the same data. While the original problem is called Primal Problem, the other is called it's Dual problem. In fact either problem can be considered as Primal and the other as the Dual. Actually both the problems can be derived from each other. The format of the Simplex Method is such that when Primal is solved, the associated Dual is also solved at the same time.

1. Important points related to formulation of Dual Problem

1. If the Primal Problem is having the objective of Maximization then the Dual will have the objective of Minimization and vice versa.

2. The unit contributions (c) of the decision variables in the Objective Function of Primal appear as constants in the right hand side of constraints of Dual.
3. Constants (b_i) at the right hand side of the constraints of Primal appear as the unit contributions of the decision variables in the Objective Function of Dual.
4. The number of decision variables in the Primal are equal to the number of constraints in the Dual.
5. The number of constraints in the Primal are equal to the number of decision variables in the Dual. While considering the number of constraints in the Primal, one must not consider the Non negativity constraint.
6. For the Primal Problem with less than or less than equal to type constraints, the Dual will have more than or more than equal to type constraints and vice versa.
7. The coefficients (a_{ij}) for the Dual decision variables in the constraints are the coefficients of the Primal decision variables in the constraints with rows and columns interchanged.
8. Any Primal Problem of Maximization should have only LESS THAN or LESS THAN EQUAL TO type Constraints. In case that is not so, then those should be converted to Less than equal to type.

Similarly any Primal Problem of Minimization should have only MORE THAN or MORE THAN EQUAL TO type Constraints. If it is not so then the constraints should be converted to More than equal to type.

To explain the above concepts regarding formulation of Dual Problem the following illustrations are going to be helpful.

Illustration 9 (Minimization type Primal problem with all the Constraints having greater than equal to type form)

Obtain the Dual of the following Primal Problem.

Minimize $Z = 6000x_1 + 4000x_2$ subject to the constraints $4x_1 + x_2 \geq 12$, $9x_1 + x_2 \geq 20$, $7x_1 + 3x_2 \geq 18$,

$10x_1 + 40x_2 \geq 40$ and x_1 & $x_2 \geq 0$

Solution:

To facilitate formation of Dual of the given problem, the following table is constructed.

		PRIMAL (Maximization problem)			
	Decision Variables	x_1	x_2	**Relation**	**RHS of Constraint**
DUAL (Maximization problem)	y_1	4	1	≥	12
	y_2	9	1	≥	20
	y_3	7	3	≥	18
	y_4	10	40	≥	40
	Relation	≤	≤	-	-
	RHS of Constraint	6000	4000	-	-

Dual problem is given as –

Maximize $Z^* = 12y_1 + 20y_2 + 18y_3 + 40y_4$

Subject to the constraints

$4y_1 + 9y_2 + 7y_3 + 10y_4 \leq 6000$

$y_1 + y_2 + 3y_3 + 40y_4 \leq 4000$

y_1 , y_2 , y_3 and $y_4 \geq 0$

[**N.B -** Following points should be noted from the above solution –

1. Primal is a problem of MINIMIZATION having Objective Function Z, while Dual is a problem of MAXIMIZATION having Objective Function Z*.
2. The number of Decision Variables in Primal is two (i.e x_1 and x_2). Hence the number of Constraints in Dual is also two.
3. The number of Constraints in Primal is four. So the number of Decision Variables in Dual is also four (y_1, y_2, y_3 and y_4)
4. In Primal the coefficients of the Decision Variables in the Objective Function are 6000 and 4000 respectively. In Dual the Right Hand Side (RHS) constants of the Constraints are formed by these two.
5. Coefficients of the Decision Variables in the Constraints of the Primal are represented column-wise in the above Table which becomes the Coefficients of the Decision Variables in the Constraints of the Dual and represented row-wise in the solution.
6. Primal has greater than or equal to (≥) type inequality Constraints while Dual has less than or equal to (≤) type inequality Constraints.
7. Number of Non negativity Constraints are same as the Number of Decision Variables in both Primal and Dual.]

Illustration 10 (Maximization type Primal problem with mixed type Constraints)

Write the Dual of the following –

Maximize $Z = 5x_1 + 10x_2$ subject to $2x_1 – 3x_2 \leq 7$, $x_1 + 2x_2 = 4$ and $x_1 , x_2 \geq 0$

Solution:

Given Primal is a problem of Maximization. So it's Dual can be written only if all the Constraints are less than or equal to type. But one of the Constraints is equal to type. Thus the same needs to be converted to less than type which is done as shown below.

$x_1 + 2x_2 = 4$ can be replaced by two inequalities $x_1 + 2x_2 \leq 4$ and $x_1 + 2x_2 \geq 4$

Again the inequality $x_1 + 2x_2 \geq 4$ can be re-written as $– (x_1 + 2x_2) \leq - 4$ or, $- x_1 - 2x_2 \leq - 4$

So the given Primal is re-written as

Maximize $Z = 5x_1 + 10x_2$

Subject to

$2x_1 – 3x_2 \leq 7$, $x_1 + 2x_2 \leq 4$, $- x_1 - 2x_2 \leq - 4$ and $x_1 , x_2 \geq 0$

		PRIMAL (Maximization problem)			
	Decision Variables	x_1	x_2	Relation	RHS of Constraint
DUAL (Minimization problem)	y_1	2	-3	$\leq$	7
	y_2	1	2	$\leq$	4
	y_3	-1	-2	$\leq$	-4
	Relation	$\geq$	$\geq$	-	-
	RHS of Constraint	5	10	-	-

Dual problem is as follows –

Minimize $Z^* = 7y_1 + 4y_2 - 4y_3$

Subject to

$2y_1 + y_2 - y_3 \geq 5$

$- 3y_1 + 2y_2 - 2y_3 \geq 10$

$y_1 , y_2 , y_3 \geq 0$

Considering $y_2 - y_3 = y_4$ the Dual problem can be written as follows

Minimize $Z^* = 7y_1 + 4y_4$

Subject to

$2y_1 + y_4 \geq 5$

$- 3y_1 + 2y_4 \geq 10$

$y_1 \geq 0$ and y_4 is unrestricted in sign

Illustration 11 (Maximization type Primal problem with mixed type Constraint and one variable has no restriction of sign)

Find the Dual program of the following LPP –

Maximize $Z = 3x_1 + 5x_2 + 7x_3$ subject to $x_1 + x_2 + 3x_3 \leq 10$, $4x_1 - x_2 + 2x_3 \geq 15$ and $x_1 , x_2 \geq 0$ & x_3 is unrestricted in sign.

Solution:

As x_3 is unrestricted in sign, to maintain the basic assumption of non-negativity of the Decision Variables in LPP, it is assumed that x_3 is the difference of two non-negative variables x_4 and x_5 i.e $x_3 = x_4 - x_5$

So the given problem is re-written as

Maximize $Z = 3x_1 + 5x_2 + 7x_4 - 7x_5$

Subject to

$x_1 + x_2 + 3x_4 - 3x_5 \leq 10$

$4x_1 - x_2 + 2x_4 - 2x_5 \geq 15$

$x_1 , x_2 , x_4 , x_5 \geq 0$

But the given problem relates to the case of Maximization. So for getting it₹s Dual, all the constraints should be of less than or equal to form. Here the 2nd constraint is of more than equal to form. It requires to be converted to less than equal to form. That is done as below.

$4x_1 - x_2 + 2x_4 - 2x_5 \geq 15$ Or, $- 4x_1 + x_2 - 2x_4 + 2x_5 \leq - 15$

Thus the Primal Problem takes the form as below.

Maximize $Z = 3x_1 + 5x_2 + 7x_4 - 7x_5$

Subject to the constraints

$x_1 + x_2 + 3x_4 - 3x_5 \leq 10$

$- 4x_1 + x_2 - 2x_4 + 2x_5 \leq - 15$

$x_1 , x_2 , x_4 , x_5 \geq 0$

The data are summarized in the Table below to facilitate formation of Dual.

		PRIMAL (Maximization problem)					
	Decision Variables	x_1	x_2	x_4	x_5	**Relation**	**RHS of Constraint**
DUAL (Minimization problem)	y_1	1	1	3	- 3	$\leq$	10
	y_2	- 4	1	- 2	2	$\leq$	- 15
	Relation	$\geq$	$\geq$	$\geq$	$\geq$	-	-
	RHS of Constraint	3	5	7	- 7	-	-

So the Dual Problem is –

Minimize $Z^* = 10y_1 - 15y_2$

Subject to the constraints

$y_1 - 4y_2 \geq 3$

$y_1 + y_2 \geq 5$

$3y_1 - 2y_2 \geq 7$

$- 3y_1 + 2y_2 \geq - 7$ Or, $3y_1 - 2y_2 \leq 7$

$y_1 , y_2 \geq 0$

The 3rd and 4th Constraints can be combined to give an equality constraint $3y_1 - 2y_2 = 7$

Thus the required Dual is

Minimize $Z^* = 10y_1 - 15y_2$

Subject to the constraints

$y_1 - 4y_2 \geq 3$

$y_1 + y_2 \geq 5$

$3y_1 - 2y_2 = 7$

$y_1, y_2 \geq 0$

2. Inter-relationship of the Optimum solutions of Primal and Dual

The solution values for the Primal can be obtained directly from the table showing optimum solution of the Dual. The methodology is described as follows –

1. The value in the Index Row or $(C_j - Z_j)$ Row corresponding to the Slack or Surplus variables (as the case may be) of the final table of the solution of Dual problem corresponds to the values of the Basic Variables, with sign changed, in the final table of Primal solution.
2. Values for the Slack/ Surplus variables of the optimal Primal solution are given by the values, with sign changed, in the Index Row under the non-basic variables of the optimum Dual solution.
3. The optimum value of the Objective Function is same for both Primal and Dual solutions.

Illustration 12

A retired person has plans to invest in shares. He has been suggested by one of his friends who plays in the share market to invest in two shares A and B which gives dividends @ 12% and 4% p.a respectively. For an investment of ₹1, the growth in the market value of the shares A and B are respectively 10 paise and 40 paise in one year. The retired person wants to invest such that the dividend income is at least ₹600 p.a and the growth of initial investment in one year is at least ₹1000.

(i) Formulate it as a Linear Programming Problem.

(ii) Write its Dual.

(iii) Solve the Dual using Simplex Method. Interpret the solution.

Solution:

(i) Let x_1 and x_2 be the number of units of the shares A and B to be purchased by the retired person. The LP can be formulated as –

Minimize $Z = x_1 + x_2$

Subject to the Constraints

$0.12x_1 + 0.04x_2 \geq 600$ (Constraint on the income from Dividend)

$0.10x_1 + 0.40x_2 \geq 1000$ (Constraint on the income from Growth)

$x_1, x_2 \geq 0$ (Non-negativity Constraint)

This is the formulated Primal Problem

(ii) To obtain the Dual, the data are summarized in the table below.

		PRIMAL (Minimization problem)			
	Decision Variables	x_1	x_2	Relation	RHS of Constraint
DUAL (Maximization problem)	y_1	0.12	0.04	$\geq$	600
	y_2	0.10	0.40	$\geq$	1000
	Relation	$\leq$	$\leq$	-	-
	RHS of Constraint	1	1	-	-

The Dual is given as –

To Maximize $Z^* = 600y_1 + 1000y_2$

Subject to the Constraints

$0.12y_1 + 0.10y_2 \leq 1$

$0.04y_1 + 0.40y_2 \leq 1$

$y_1 \ , \ y_2 \geq 0$

(iii) To solve the Dual using Simplex Method, we introduce non-negative Slack Variables (S_1 & S_2) to rewrite the Constraints as follows –

$0.12y_1 + 0.10y_2 + S_1 = 1$

$0.04y_1 + 0.40y_2 + S_2 = 1$

So the Dual LPP in standard form is written as

Minimize $Z^* = 600y_1 + 1000y_2 + 0.S_1 + 0.S_2$

Subject to the Constraints

$0.12y_1 + 0.10y_2 + S_1 = 1$

$0.04y_1 + 0.40y_2 + S_2 = 1$

$y_1 , y_2 , S_1 , S_2 \geq 0$

Considering $y_1 = y_2 = 0$, the Initial Feasible Solution is $S_1 = 1$ & $S_2 = 1$. This is shown in the following Table.

Table – 1 showing Initial Feasible Solution

C_j			600	1000	0	0	Min. Ratio	
C_B	Basic Variables	Solution Values (X_B)	y_1	y_2	S_1	S_2		
0	S_1	1	0.12	0.10	1	0	1/0.10 = 10	
0	S_2	1	0.04	**0.40***	0	1	1/0.04 = 2.5	Key Row
$(Z_B)^* = 0$	-	$(Z_j)^*$	0	0	0	0	y_2 enters & S_2 departs	
		$C_j - (Z_j)^*$	600	1000	0	0		
				Key Col.				

Table – 2 showing Improved Solution

C_j			600	1000	0	0	Min. Ratio	
C_B	Basic Variables	Solution Values (X_B)	y_1	y_2	S_1	S_2		
0	S_1	0.75	0.11^*	0	1	- 0.25	0.75/0.11 = 6.82	Key Row
1000	y_2	2.5	0.10	1	0	2.5	2.5/0.10 = 25	
$(Z_B)^* = 2500$	-	$(Z_j)^*$	100	1000	0	2500	y_1 enters & S_1 departs	
		$C_j - (Z_j)^*$	500	0	0	- 2500		
			Key Col.					

Table – 3 showing Optimal Solution

C_j			600	1000	0	0	Min. Ratio
C_B	Basic Variables	Solution Values (X_B)	y_1	y_2	S_1	S_2	
600	y_1	6.82	1	0	9.1	- 2.27	
1000	y_2	1.82	0	1	- 0.91	2.73	
$(Z_B)^* = 5912$	-	$(Z_j)^*$	600	1000	4550	1368	
		$C_j - (Z_j)^*$	0	0	- 4550	- 1368	

As all the values in the Index Row are zero or negative, optimality condition has been attained. The optimum solution is –

Maximum value of $Z^* = 5912$ and it corresponds to $y_1 = 6.82$ and $y_2 = 1.82$

Here the values of y_1 and y_2 indicates the marginal worth of one unit of Shares A and B respectively. Also the shadow prices (that is the optimum values of the dual variables) are the coefficients of the surplus variables in the Objective Function of the Primal problem.

Also the values, with sign changed, in the Index Row of the optimum Dual solution corresponding to the non-basic variables (that is S_1 and S_2) indicate the optimum values of the decision variables (x_1 and x_2) of the Primal problem. Thus the optimum number of Shares A to be purchased (x_1) = 4550 and that of Share B to be purchased (x_2) = 1368

Also optimum values of Objective Function in both Primal and Dual should be same i.e $(Z)_{Minimum} = (Z^*)_{Maximum} = 5912$

[Above interpretation of optimum Dual Solution can be cross checked by solving the Primal using Simplex Method or otherwise]

3. Characteristics of the Dual Problem

1. Dual of a Dual is the Primal.
2. If either of the Primal or Dual has a solution, then the other should definitely give a solution. Also optimum values of both the solutions are equal.
3. If any of the two types of problems has Infeasible solution, then the Objective Function of the other should have Unbounded optimal solution and vice versa.
4. The value of the Objective Function for any feasible solution of the Primal is less than the value of the Objective Function for any feasible solution of the Dual. But the optimum values of the objective functions of both are same.
5. If the Primal has a feasible solution but the Dual does not have, then the Primal will not have a finite optimum solution and vice versa.

EXERCISE

A. Theoretical Questions:

Multiple Choice Questions

1. A constraint in an L.P. Model restricts
   a. Value of the Objective Function.
   b. Values of the Decision Variables
   c. Use of the available resources
   d. All the above

2. In graphical method of solution of LPP if the Iso-cost line coincide with a side of the Feasible Region then we get –
   a. Unique optimum solution.
   b. Unbounded optimum solution.
   c. No feasible solution.
   d. Infinite number of optimum solutions.

3. A feasible solution of LPP –
   a. Must satisfy all the constraints simultaneously.
   b. Need not satisfy all the constraints, only some of them.
   c. Must be a corner point of the feasible region
   d. All the above.

4. The Objective Function of a LPP is $Z = 3x_1 + 2x_2$. If $x_1 = 10$ and $x_2 = 5$ then the value of Z is –
   a. 35
   b. 40
   c. 45
   d. 50

5. Multiple solution exist in a Linear Programming problem when –
   a. One of the constraints is redundant
   b. Objective Function is parallel to one of the constraints
   c. Two constraints are parallel
   d. All of the above

6. The linear function of the variables which is to be optimized is called –
   a. Constraints
   b. Objective Function

c. Decision variables

d. None of the above

7. If the value of the Objective Function can be increased or decreased indefinitely then the solution is called –

a. Unbounded

b. Bounded

c. Infeasible

d. None of the above

8. The first step in formulating a LPP is –

a. Identify the upper and lower boundaries of the decision variables

b. State the constraints as linear combinations of the decision variables

c. Understand the problem

d. Identify the Decision Variables

9. The best use of Linear Programming is to find the optimal use of –

a. Manpower

b. Material

c. Money

d. All of the above

10. Which of the following is assumption of Linear Programming Model?

a. Divisibility

b. Proportionality

c. Additivity

d. All of the above

11. Non-negativity condition of Linear Programming implies –

a. A positive coefficient of variables in Objective Function.

b. A positive coefficient of variables in any constraint.

c. Non-negative value of resource.

d. None of the above.

12. If the constraints of a Linear Programming problem are $x_1 + x_2 \leq 1$, $3x_1 + x_2 \geq 3$ and x_1 , $x_2 \geq 0$ then –

a. There are two feasible regions

b. There are infinite feasible regions

c. No feasible region

d. None of the above

13. For any LPP the intermediate solutions must be checked by substituting them back into the

a. Objective Function

b. Constraints

c. Either of (a) and (b)

d. This is not required.

14. The feasible solution of any LPP should belong to –

a. Both first and second quadrant

b. Only first quadrant

c. Only second quadrant

d. Both first and third quadrant

15. The true statement related to the graphs of $3x_1 + 2x_2 \leq 5$ and $6x_1 + 4x_2 > 10$ is –

a. Both the graphs are disjoint.

b. Both contain the point (1,1)

c. Both (a) and (b) above are true

d. Both (a) and (b) are not true simultaneously

16. In which quadrant the bounded region of the inequalities $x_1 + x_2 \leq 1$ and $x_1 - x_2 \leq 1$ is situated?

a. First and third

b. Second and third

c. First and second

d. All the four quadrants

17. Objective function of LPP is –

a. A relation between the variables

b. A function to be optimized

c. A constraint

d. None of the above

18. The optimal value of the Objective Function is attained at the poinrs

a. Given by intersection of inequations with axes only

b. Given by intersection of inequations with x axis only

c. Given by intersection of inequations with y axis only

d. Given by corner points of the feasible region.

19. If the constraints in a Linear Programming problem are changed then –

a. The problem is to be re-evaluated.

b. Solution is not defined

c. The Objective Function has to be modified.

d. The change in constraints is to be ignored.

20. The constraints $y - x \leq 1$, $3y - x \leq 9$ and $x, y \geq 0$ are defined on

a. Bounded feasible space

b. Unbounded feasible space

c. Redundant space

d. None of the above.

21. Which of the terms is not used in Linear Programming?

a. Slack variables

b. Objective function

c. Concave region

d. Feasible region

22. The area of the Feasible Region of the constraints $3x_1 + x_2 \geq 3$, $x_1 \geq 0$ and $x_2 \geq 0$ is –

a. Bounded

b. Unbounded

c. Convex

d. Concave

23. For the LPP, Minimize $Z = x + y$ subject to the constraints $5x + 10y \leq 0$, $x + y \geq 1$, $y \leq 4$, $x \geq 0$ and $y \geq 0$

a. There is a bounded solution

b. There is no solution

c. There are infinite solutions

d. None of the above

24. In a Linear Programming Problem –

a. Objective Function is linear.

b. Constraints are linear.

c. Both Objective Function and Constraints are linear.

d. None of the above

25. Constraints mean –

a. Limitations are expressed in the form of mathematical inequalities or equalities.

b. Assumption

c. Goal to be achieved

d. None of the above

26. The region which satisfies all the constraints of LPP is known as –

a. Phisible region

b. Convex region

c. Feasible region

d. Concave region

27. In LPP while drawing the graph, y values on x axis are always –

a. 1

b. 0

c. -1

d. All of the above

28. The set of decision variables which satisfies all the constraints of LPP is called –

a. Solution

b. Basic solution

c. Feasible solution

d. None of the above

29. The value of the Objective Function is maximum under linear constraints –

a. At the centre of Feasible Region

b. At the origin

c. At a vertex of the Feasible Region

d. At the vertex of the Feasible Region which is farthest from the origin.

30. A solution which optimizes the Objective Function is called –

a. Solution

b. Basic solution

c. Feasible solution

d. Optimal solution

Answers

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
c	b	a	b	b	b	a	d	d	d	c	c	d	b	a
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
d	b	d	a	b	c	b	a	d	a	c	b	c	d	d

State True or False

1. The feasible region of the constraints $5x + y \leq 100$, $x + y \leq 60$, $x \geq 0$ and $y \geq 0$ does not contain the point (60,0) on its boundary.
2. In the optimal table of a LPP when there exists a non-basic variable with zero value of relative profit [i.e $(C_j - Z_j)$], then an alternate solution to the problem is possible.
3. The LPP, Maximize $Z = 60x + 50y$ subject to $x + 2y \leq 40$, $3x + 2y \leq 60$, $x, y \geq 0$ has unbounded feasible solution.
4. Artificial variables are introduced to maintain non-negativity of Surplus variables in the Initial Basic Feasible Solution of LPP.
5. Every Linear Programming Problem has a unique solution.
6. If (3,3), (20,3), (20,10), (18,12) and (12,12) are the corner points of the feasible region of the Linear Programming Problem – Maximize $Z = 2x_1 + 3x_2$ then maximum value of the Objective Function is 70.
7. For the LPP – Minimize $Z = 5x + 3y$, Iso-cost lines are a family of straight lines having slope (- 5/3).
8. Simplex Method does not provide any indication about infeasibility of solution of a LPP.
9. Linear Programming can provide solution of the situations even if management have conflicting goals.
10. Linear Programming is used for solving problems related to portfolio selection.
11. A redundant constraint is one that does not affect the feasible solution region.
12. In Simplex Method, the optimality condition is said to be arrived when each and every entry in the Index Row of the table is negative.
13. An Artificial Variable has an infinitely large positive contribution towards the Objective Function for a problem of maximization.
14. For the LPP – Maximize $Z = 3x + 6y$ subject to $x + y \leq 20$, $2x + y \leq 30$, $x \leq 35$, $x \geq 0$ and $y \geq 0$ the redundant constraint is $x \leq 35$
15. Linear Programming Problems of maximization have multiple optimum solution if the Iso-profit lines are parallel to any one or more of the Constraint lines.
16. Pivot column is decided on the basis of the maximum positive entry in the Index Row of a Simplex Table for a maximization problem of LP.
17. Airline Routing problems cannot be solved by Simplex Method.
18. Situation of degeneracy occurs for a LPP when one or more of the basic variables become zero.
19. 3 of the points (1,1), (–1,1), (1,–1), (–1,–1), (–2,1), (2,–1), (–2,–1) & (2,1) satisfy the constraint $2x - 3y > -5$

20. The maximum value of $Z = 3x + 4y$ subject to the constraints $x + y \leq 4$ and $x, y \geq 0$ is 12.
21. The minimum value of $Z = 3x + 2y$ subject to the constraints $x + y \geq 8$, $3x + 5y \leq 15$ & $x, y \geq 0$ is 6
22. If a feasible region is unbounded then the LPP has no solution.
23. Entries in the Z_j row of a Simplex Table corresponding to the non-basic variables indicate decrease in the value of Objective Function for one unit increase in the non-basic variables.
24. The position of the origin and the point (2,–3) in the graph of the constraint $2x - 3y < 5$ can be given as – Origin inside and (2,–3) outside the region.
25. Mathematical model of Linear Programming is important because decision makers prefer to work with formal models.
26. Alternative solution exists in a LPP when two constraint lines are parallel.
27. Conditions of certainty is assumed to exist in case of Linear Programming.
28. For solution of a Linear Programming problem of minimization by Simplex optimality is said to achieve when all the entries in the Index Row of the table are either zero or positive.
29. One kind of cake requires 300 gms. of flour and 15 gms. of fat. Another kind of cake requires 150 gms of flour and 30 gms. of fat. If 7.5 Kgs. of flour and 600 gms. of fat are available then the constraint on the flour availability can be expressed as $2x + y \leq 50$, where x and y are the quantities of the two types of cakes to be produced for maximization of production.
30. Simplex Method was developed during World War I to utilize the scarce resources properly.

Answer

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T	T	F	T	F	F	T	F	F	T	T	F	F	T	T
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
T	F	T	F	F	F	F	T	T	F	F	T	T	T	F

Fill in the Blanks

1. In Simplex Method of solution of LPP, _____ variables are introduced to convert less than type inequalities into equations.
2. A basic solution with m equations and n variables has ____ number of variables equal to zero when n > m.
3. A basic feasible solution is a basic solution whose variables are ______.
4. In any LPP every ____ point of the Convex Set of Feasible Solutions is a ______ solution of the problem.
5. The Objective Function of a Linear Programming Problem is maximized or minimized at the ______ solution.
6. A linear programming problem has a well-defined Objective Function which is linear and which has to be _____.
7. The constraints in a Linear Programming model arises due to limitation of ______.
8. The solution of a Linear Programming Problem indicates the right combination of _____ variables.

9. The set of variables appear in the Basic Variable column of a Simplex table is known as ____.
10. The variables which does not appear in the Basis is known as ______ variables.
11. In LPP the number of decision variables and constraints must be ______.
12. ____ element of the Simplex Table is the element at the junction of Pivot Row and Pivot Column.
13. To convert a more than equal to type constraint into an equation, introduction of a ____ variable and an Artificial variable is necessary while solving LPP by Simplex Method.
14. Contribution of Slack or Surplus variable towards Objective Function is ____ always.
15. If the constraints of a LPP are $2x + y \leq 50$, $x + 2y \leq 40$, $x \geq 0$ and $y \geq 0$, then the feasible region lies in the ____ quadrant.
16. Linear Programming facilitates ___ use of productive factors.
17. For the LPP, Minimize $Z = 4x + 5y$ the coordinates of the corner points of the bounded feasible region are (10,10), (20,5), (2,17), (16,11) and (17,5). The minimum value of Z is ______.
18. The corner points of feasible region of a LPP are (0,15), (15,15), (25,25), (10,35) and (10,0) If the Objective Function is $Z = px + qy$ then the condition for which Z will be maximum at both the points (25,25) and (10,35) is _____.

Answer:

1	**Slack**	2	**n - m**
3	**Non-negative**	4	**Extreme**
5	**Optimal**	6	**Optimized**
7	**Resources**	8	**Decision**
9	**Basis**	10	**Non-basic**
11	**Finite**	12	**Key**
13	**Surplus**	14	**Zero**
15	**First**	16	**Optimal**
17	**90**	18	**3p = 2q**

Short essay type questions

1. What is meant by Mathematical Model?
2. Why the term slack is used in case of the variable used for converting a less than equal to type constraint inequality into an equation?
3. To convert more than or more than equal to type constraint inequalities into equations Surplus and Artificial variables are used. What is the purpose of using Artificial variable?
4. Two methods of solving LPP are in use – Graphical and Simplex. Which one is more versatile & why?
5. Iso-cost lines are used to solve minimization problems of LPP graphically. What is meant by the term Iso-cost?

6. Name different types of Management problems which can be solved by LP technique.
7. Why the term linear is used in relation to a LPP?

Essay type questions

1. What are the requirements of Linear Programming?
2. What are the advantages and limitations of Linear Programming Models?
3. What are the different application areas of Linear Programming?
4. Define the terms – Decision Variables, Objective Function, Constraints.
5. What do you understand by a Linear Programming Problem? Give formal definition of Linear Programming.

B. Numerical Questions

Comprehensive Numerical Problems

1. An investor is well aware of the fact that maximization of total return on investment is best possible if the help of Linear Programming technique is taken. For that he has employed you and provided with the following information about his various activities of money making.

   Activity A_1 – Invest in Bank Fixed Deposit schemes, Activity A_2 – Invest in Government Bonds, Activity A_3 – Invest in Midcap Mutual Funds and Activity A_4 – Invest in Equity linked Mutual Funds.

   Total amount to be invested – ₹ 5 lakhs

   To avoid excessive investment, not more than 50% of the total should be invested in Government Bonds and Midcap Mutual Funds.

   Investment in Bank Fixed Deposits is very conservative way of money making, while Investment in Equity linked Mutual Funds is very speculative. To avoid excessive speculation, at least ₹ 1 must be invested in Fixed Deposits for every ₹ 3 invested in Equity linked Mutual Funds.

   Data on the ROI of the different activities are as follows –

Activity	A_1	A_2	A_3	A_4
Return on Investment,	4%	5.5%	7%	10%

   To solve the problem the investor has provided you with software. But you have to arrange for formulating the mathematical model of the problem and subsequent data feeding to use the software. Explain clearly your first step.

2. For a manufacturing company the following data are relevant to its products A and B

Description	Product A	Product B
1) Selling price per unit in Rupees	200	240
2) Cost per unit in Rupees		
2.1) Direct materials	45	50
2.2) Direct wages in		

Description	Product A	Product B
Department 1	16	20
Department 2	22.50	15.50
Department 3	10/-	30/-
2.3) Variable Overhead	6.50/-	11.50/-
3) Hours required per unit in		
Department 1	8	10
Department 2	10	6
Department 3	4	12

Also it is given that the number of employees in the Departments 1, 2 and 3 are respectively 20, 15 and 18 as well as the maximum available hours per employee per week for each of the three departments is 40. Formulate the linear programming problem so that the contribution of the company is maximized.

3. A company buying scrap metal has two types of scrap available to them. The first type of scrap has 20% of Metal A, 10% impurity and 20% of Metal B by weight. The second type of scrap has 30% of Metal A, 10% impurity and 15% of Metal B by weight. The company requires at least 120 kgs. of Metal A, at most 40 kgs. of impurity and at least 90 kgs. of Metal B. The price for the two scraps are ₹ 200 and ₹ 300 per kg. respectively. Determine the optimum quantities of the two scraps to be purchased by the company so that the requirements of the two metals and the restriction on impurity are satisfied at minimum cost. Use graphical method.

4. An engineering company dealing with Combustion Equipment and Furnaces is planning its advertising strategy. They have two monthly industrial magazines under consideration. The first magazine has a reach of 2000 potential customers per advertisement and the second magazine has a reach of 3000 potential customers per advertisement. Respective cost per advertisement of the two magazines are ₹ 6000 and ₹ 9000 and the firm has a monthly budget of ₹ 1 lakh. There is an important requirement that the total reach for the business group having annual turnover below ₹ 20 Crores must not exceed 3000 potential customers. The reach of the two magazines having such business group customers are respectively 300 and 150 potential customers per advertisement. How many times the company should advertise in the two magazines to maximize the total reach? Solve graphically.

5. A Factory manufactures 3 products which are processed through 3 different production stages. The time required to manufacture one unit of each of the three products and the daily capacity of the stages are given in the following table:

Stage	Time per unit in minutes			Stage capacity (minutes)
	Product 1	Product 2	Product 3	
1	1	2	1	430
2	3	-	2	460

Stage	Time per unit in minutes			Stage capacity (minutes)
	Product 1	Product 2	Product 3	
3	1	4	-	420
Profit per unit (₹)	3	2	5	

Set the data in a Simplex Table and find the optimal solution.

6. Obtain the Dual from the following Primal.

   Minimize $Z = x_1 - 3x_2 - 2x_3$

   Subject to the Constraints

   $3x_1 - x_2 + 2x_3 \leq 7$

   $2x_1 - 4x_2 \geq 12$

   $-4x_1 + 3x_2 + 8x_3 = 10$

   $x_1 \geq 0$, $x_2 \geq 0$ and x_3 is unrestricted in sign.

7. A company makes three products X, Y and Z using the raw materials A, B and C, Requirement of raw materials for each of the products X, Y and Z are given below.

   Each of Product X requires 1 unit of raw material A, 2 units of B and 2 units of C.

   Each of Product Y requires 2 units of raw material A, 1 unit of B and 5 units of C.

   Each of Product Z requires 1 unit of raw material A, 4 units of B and 1 unit of C.

   Find the optimum product mix when it is given that each unit of X, Y and Z gives profit of ₹40, ₹25 and ₹50 respectively. Write the Dual of the given problem. Using the optimum solution of the Primal problem, write the solution of the Dual and interpret it.

Answer:

1. Mathematical formulation of the given LPP would be the first step and that is given as follows.

   Maximize $Z = 0.04x_1 + 0.055x_2 + 0.07x_3 + 0.10x_4$ (x_i is the amount invested in Activity A_i and i = 1, 2, 3, 4)

   Subject to the constraints

   $x_1 + x_2 + x_3 + x_4 \leq 500000$, $-0.5x_1 + 0.5x_2 + 0.5x_3 - 0.5x_4 \leq 0$, $3x_1 - x_4 \geq 0$ & $x_1, x_2, x_3, x_4 \geq 0$

2. Maximize Z = 100x + 115y (x and y are the number of Products A and B are to be manufactured)

   Subject to the constraints

   $8x + 10y \leq 800$, $10x + 6y \leq 600$, $4x + 12y \leq 720$ & $x, y \geq 0$.

3. The solution is infeasible.

4. The problem has multiple optimum solution with maximum value of Objective Function = 100000/3.

5. Maximum Profit = ₹ 1350

   No. of units of Product 1 to be produced = 0

   No. of units of Product 2 to be produced = 100

   No. of units of Product 1 to be produced = 230

6. Maximize $Z^* = -7y_1 + 12y_2 + 10y_3$

   Subject to the constraints

   $-3y_1 + 2y_2 + 4y_3 \le 1$

   $y_1 - 4y_2 + 3y_3 \le -3$

   $2y_1 - 8y_3 = 2$

   $y_1 \ge 0$, $y_2 \ge 0$ and y_3 is unrestricted in sign

7. Optimum Product Mix is X = 20 Units, Y = 0 and Z = 5 Units and Maximum Profit = ₹ 1050

   Dual is given as

   Minimize $Z^* = 36P + 60Q + 45R$

   Subject to the constraints

   $P + 2Q + 2R \ge 40$

   $2P + Q + 5R \ge 25$

   $P + 4Q + R \ge 50$

   $P, Q \& R \ge 0$

   From the optimum solution of the Primal problem, the solution of the Dual is given as P = 0, Q = 10 & R = 10 and $Z^* = 1050$

   Final table of the Primal problem indicates that the marginal value of the raw material A = 0, B = ₹10 per unit and C = ₹10 per unit respectively. Thus instead of making the products X, Y and Z and selling, if the raw materials A, B and C are sold at the above rates then also the same contribution of ₹1050 can be achieved.

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